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LETTER TO THE EDITOR

On the admissible Lorentz group representations in unique-mass, unique-spin relativistic wave equations

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Abstract. Considering the infinite class of relativistic wave equations wherein the transformation property of the wavefunction involves up to four (unspecified) inequivalent irreducible representations (IIRs) of the Lorentz group occurring with arbitrary multiplicities, we have investigated the general question as to what combinations of IIRs would be admissible and with what multiplicities, if it were required that the equations have solutions only for a single (unspecified) spin and mass and be inequivalent to any simpler equation. It is found that the possibilities are quite limited. Our main results are presented here.

The discovery of inconsistencies in practically all the known theories of higher spin particles interacting with external fields (see, for example, Johnson and Sudarshan 1961, Velo and Zwanziger 1969, Wightman 1978, Mathews *et al* 1980) and more recently, certain developments in supergravity (Deser and Witten 1981, Siegel 1981, Duff and van Nieuwenhuizen 1980) have given an impetus to the search for new relativistic wave equations with a richer structure than the familiar ones (Capri 1969, 1972, Glass 1971, Fisk and Tait 1973, Hurley and Sudarshan 1975, Khalil 1977, Cox 1982a, b). The extra structure is introduced by permitting more complex transformation properties for the particle field ψ—allowing the Lorentz group representation

$$S(\Lambda) \sim \Sigma_{\otimes}\, \alpha_\tau D^{(\tau)}(\Lambda) \tag{1}$$

according to which ψ transforms to contain *repeated* irreducible representations $D^{(\tau)}$ (i.e. have multiplicity $\alpha_\tau > 1$ for some IRs $D^{(\tau)}$), or to contain spins exceeding the desired physical spin s which the wave equation

$$-\mathrm{i}\beta^\mu \partial_\mu \psi + m\psi = 0 \tag{2}$$

is to single out. However, the construction of such equations, starting with various specific choices of $S(\Lambda)$, have resulted in much fruitless effort: most of the new equations have turned out to be equivalent to simpler known ones, though looking more complicated through the presence of additional field components having no essential role. (We refer to such components as barnacles, following Hurley and Sudarshan 1975.)

This situation (besides the intrinsic interest of the problem) makes it pertinent and timely to raise the following general question. What combinations of IRs τ and

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multiplicities α_τ (out of the infinite variety that faces one at the outset) will support field equations which

(i) are assured to be inequivalent to simpler equations, begin free of barnacles, and

(ii) yield a unique spin and mass, with no degeneracy?

The conditions arising from (i) on the matrices β^μ have been determined by Khalil (1978) while those resulting from (ii) have long been known (Harish-Chandra 1947, Gel'fand *et al* 1963); nevertheless, it is only now that a question of such generality as the above one is being tackled. We have succeeded in determining to a large extent the implications of these conditions for cases where up to four (unspecified) IRs are allowed with arbitrary multiplicities in $S(\Lambda)$. We consider that this constitutes a significant advance in the decades-old field of relativistic wave equations. Our main results are summarised in this letter.

Lorentz invariance of equation (2) implies that matrix elements of β^μ in the canonical basis (labelled by the IR label τ and the quantum numbers j, λ of $\boldsymbol{J}^2$ and J_z) factorise as

$$\langle\tau' a' j'\lambda'|\beta^\mu|\tau a j\lambda\rangle = c_{a'a}^{(\tau'\tau)} g^{\mu(\tau'\tau)}(j'\lambda'; j\lambda) \tag{3}$$

where the second factor is a Lorentz group *CG* coefficient and $c_{a'a}^{(\tau'\tau)}$ is a reduced matrix element of arbitrary value. (The label a identifies a particular one of the α_τ IRs $D^{(\tau)}$.) All the c's associated with a particular pair of IRs (τ', τ) form an $\alpha_{\tau'}\times\alpha_\tau$ block $C^{(\tau'\tau)}$ and these blocks (for all the τ', τ occurring in $S(\Lambda)$) together make up the 'skeleton matrix' C associated with β^μ. Of special interest is β^0, which has a block diagonal form. Each block, associated with a particular spin j, is a direct product $\beta^0{}_{(j)}\otimes I_{(j)}$ of the *spin block* $\beta^0{}_{(j)}$ with a unit matrix of dimension $(2j+1)$; and $\beta^0{}_{(j)}$ itself is obtainable from C by multiplying each $C^{(\tau'\tau)}$ by $g_j^{(\tau'\tau)}\equiv g^{0(\tau'\tau)}(jj; jj)$ and deleting all those blocks for which either τ' or τ does not contain spin j. The mathematical conditions into which the requirements (i) and (ii) translate can now be stated as follows.

(i) The two submatrices of C, one consisting of the α_τ rows, and the other, of the α_τ columns associated with τ, must be of rank α_τ (for every τ).

(ii) The spin block $\beta^0{}_{(s)}$ pertaining to the unique physical spin s must have just one pair of non-zero eigenvalues, $+1$ and -1. Apart from these, all eigenvalues of all $\beta^0{}_{(j)}$ must be zero. This structure implies but is not wholly implied by the Harish-Chandra condition that the minimal equation of β^0 must have the form

$$(\beta^0)^{l+2} = (\beta^0)^l. \tag{4}$$

By exploiting certain basic but little-used aspects of matrix theory, we have been able to determine the implications of the above requirements on the skeleton matrix C and to obtain thus a number of useful results of a rather general nature regarding the admissibility of various combinations of IRs and multiplicities. The following are among the most interesting of these.

(1) With the number of inequivalent irreducible representations (IIRs) restricted to two, our requirements admit just two possibilities (Mathews *et al* 1981):

$$S(\Lambda)\sim(s, 0)\oplus(s-\tfrac{1}{2}, \tfrac{1}{2}) \text{ and } S(\Lambda)\sim(s, 0)\oplus(s+\tfrac{1}{2}, \tfrac{1}{2}).$$

(2) With three IIRs, $\tau_1=(m, n)$ and τ_2, τ_3 belonging to the set $(m+\frac{1}{2}\varepsilon, n+\frac{1}{2}\varepsilon')$, $\varepsilon, \varepsilon'$ being independently $+1$ or -1: (a) Irrespective of the $D^{(\tau)}$, multiplicities related by $\alpha_1=\alpha_2+\alpha_3$ are not admissible. This result rules out the spin-0 equation based on $S(\Lambda)\sim 2(\frac{1}{2}, \frac{1}{2})\oplus(1, 1)\oplus(0, 0)$, proposed by Cox (1982b). In fact, one sees on inspection

that the equation is barnacled and equivalent to the Kemmer equation (Kemmer 1939). With parity invariance as an additional requirement, there are two possible types of representations to be considered—one with τ_1 self-conjugate ($\tau_1 = \dot{\tau}_1$) and τ_2 and τ_3 mutually conjugate ($\tau_2 = \dot{\tau}_3$), and the other with all IRs self-conjugate. In the former class, no $S(\Lambda)$ in which any $\alpha_i \neq 1$ is consistent with (i) and (ii); the only allowed case is the Kemmer equation for spin 1. In the latter class, the only equation possible with $\alpha_1 = \alpha_2$ is the Hagen–Singh spin-1 equation involving $(\frac{1}{2}, \frac{1}{2})$, $(1, 1)$ and $(0, 0)$ with all $\alpha_i = 1$. But with $\alpha_1 = \alpha_2 + 1$ ($= \alpha$, say) a whole set of equations based on $S(\Lambda) \sim \alpha(\frac{1}{2}, \frac{1}{2}) \oplus (\alpha - 1)(1, 1) \oplus \alpha(0, 0)$ is available ($\alpha = 2, 3 \ldots$). The degree $(l+2)$ of equation (4) in this case is $(2\alpha + 1)$ if s is chosen to be 0 and $(2\alpha + 3)$ for $s = 1$. Equations of this class have not been considered in the literature before. Equations involving representations with $\alpha_1 - \alpha_2 \geq 2$ could exist but with the minimal degree of β^0 not less than 7.

(3) With four IIRs and parity invariance required in addition to (i) and (ii), no combination of linked IRs (m_1, n_1), (m_2, n_2), (m_3, n_3), (m_4, n_4) is admissible other than the following.

(a), $\alpha_1(\frac{1}{2}, \frac{1}{2}) \oplus \alpha_2(1, 1) \oplus \alpha_3[(1, 0) \oplus (0, 1)]$, with either $\alpha_1 = \alpha_2 = \alpha_3 = 1$ ($S = 0$ only) or $\alpha_2 < \alpha_1$, $\alpha_3 \leq \alpha_1$ and $\alpha_1 < \alpha_2 + \alpha_3$ ($s = 0$ or 1).

(b), $\alpha_1(\frac{1}{2}, \frac{1}{2}) \oplus \alpha_2(0, 0) \oplus \alpha_3[(1, 0) \oplus (0, 1)]$ with $\alpha_2, \alpha_3 < \alpha_1 \leq (\alpha_2 + \alpha_3)$, ($s = 0$ or 1). One of the new spin-0 equations proposed by Cox belongs to this class ($\alpha_1 = 2, \alpha_2 = \alpha_3 = 1$).

(c), $\alpha_1(0, 0) \oplus \alpha_2(\frac{1}{2}, \frac{1}{2}) \oplus \alpha_3(1, 1) + \alpha_4(\frac{3}{2}, \frac{3}{2})$ with $\alpha_1 \leq \alpha_2 \leq (\alpha_1 + \alpha_3)$ and either $\alpha_4 < \alpha_3 < (\alpha_2 + \alpha_4)$, ($s = 0, 1$ or 2) or $\alpha_3 = \alpha_4 = 1$ ($s = 2$ only).

(d), $\alpha_1[(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)] \oplus \alpha_2[(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})]$. In this class, equations for $s = \frac{3}{2}$ with $\alpha_1 = 1$ can be found with any α_2, the minimal degree of β^0 being $(\alpha_2 + 3)$. The cases $\alpha_2 = 1$ and 2 yield the Rarita–Schwinger (1941) and Glass (1971) equations respectively. Cases with $\alpha_1 > 1$ have not so far been amenable to complete classification. Also spin-$\frac{1}{2}$ equations exist which must have $\alpha_1 = \alpha_2 - 1$ if the minimal degree of β^0 is restricted to 4. The Khalil (1977) equation is the simplest member of this class, with $\alpha_1 = 1$, $\alpha_2 = 2$. (The Santhanam-Tekumalla (1974) equation, which has $\alpha_1 \neq \alpha_2 - 1$, is barnacled.) Others with minimal degree exceeding 4 are not ruled out. For example, with $\alpha_1 = 1$ and arbitrary α_2, the minimal degree is $(\alpha_2 + 2)$.

The above results, for the first time, give general guidelines as to the regimes of multiplicities and IRs wherein one may look for really new equations, for example in attempting to escape the inconsistencies which beset the familiar equations. It may be noted that for spins > 2, one has to allow for more than 4 IIRs. Some of the results can be generalised to such cases too. The proofs of our results, which are rather involved though essentially elementary, will be presented in a separate detailed paper.

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